Journal
Of the
Oriental Institute
M.S. University of Baroda

ISSN: 0030-5324
UGC CARE Group 1

P´AL TYPE (0;1) INTERPOLATION ON THE ULTRASPHERICAL ABSCISSAS

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1 Abstract

We study about the P´al-type interpolation on the roots of Ultraspherical polynomials along with the boundary (Hermite) conditions placed at the endpoints of the finite interval [-1,1], which gives a simultaneous approximation of a differentiable function and the function's derivative. The order of convergence depends only on the smoothness of the function. In this paper, we study about interpolation on polynomials (along-with the Hermite boundary conditions) where the nodes are the zeroes of Ultraspherical polynomials $P_{n-1}^{(k+1)}(x)$ and $P_{n-1}^{(k+1)\prime}(x)$ respectively. Here $P_n^{(k)}(x)$ represents the Ultraspherical polynomial of degree n. Our focus is to find the existence, uniqueness, explicit representation, and order of convergence of the interpolatory polynomials.

Keywords: P´al-type interpolation; Ultraspherical polynomials; Lagrange interpolation; Fundamental polynomials; Hermite-type boundary conditions;
Explicit form; Order of convergence
2020 Mathematical Sciences Classification: 41A10, 97N50.

2 Introduction

In P´al-type interpolation, we interpolate the function values on the zeros of polynomial, w(x), while interpolation of the first derivative's values is done on the roots of w'(x). Whereas in the inverse P´al-type interpolation, the roles of w(x) and w'(x) are interchanged, i.e. the values of derivative are interpolated at the roots of w(x) while the function values are interpolated at the roots of w'(x). In easy words we can say that the derivative of our interpolational polynomial interpolates the functions derivative values at the roots of w'(x). This inverse type of interpolation aroused the question that under what conditions we can work out the P´al-type interpolation on a simultaneous approximation of a differentiable function and its derivative. In 2001, Lenard[3] interpolated the function values at the zeros of the polynomial $P_{n-1}^{(k+1)}(x)$ while the first derivative values were interpolated at the zeros of the polynomial $P_n^{(k)}(x)$ along with the Hermite (boundary) conditions on the interval [-1, 1]. Again in 2004, Lenard[4] did a study on the modified P´al-type interpolation problem on

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Of the
Oriental Institute
M.S. University of Baroda

ISSN: 0030-5324
UGC CARE Group 1

the zeroes of the Laguerre abscissas. Later, two pairwise disjoint sets, which were the zeros of $P_n(x)$ and $\pi_n(x)$ with two new additional conditions were discussed, while Srivastava[6] in 2014 discussed about the interpolation process on the roots of the Hermite polynomials on infinite interval. Further in 2013, Lenard[5] worked on the Pa´l-type interpolation problem with two sets of nodes, where one consisted of the zeros of a polynomial $P_n(x)$ (of degree n), while the constituents of the other one are the zeros of $P'_n(x)$, so we can say that two different interpolatory conditions were prescribed simultaneously. In 2013, the (0;1) P´al-type interpolation on mixed Tchebychef polynomials was investigated. Further, in 2017, the weighted (0,2) interpolation polynomials on the roots of all classical orthogonal polynomials was discussed. In 2019, Yamini Singh and R.Srivatava[8] solved (0,1;0) interpolation problem with special kind of boundary conditions on the roots of the Ultraspherical polynomials.

In 1983 inverse Pal type interpolation problem was considered on the roots of integrated Legendre's polynomial. Then the func- tion values were consiered by toher authors to be interpolated on the zeros of the polynomial $P -n(k+1)(x)$ and the first derivative's values were interpolated on the zeros of the polynomial $P_n^{(k)}(x)$ with the Hermite conditions on the interval [-1,1].

Xie[10] studied the convergence of interpolation process for $k = 0$,

$$|f(x) - R_{2n+1}(x;f)| = w(f^{(r)};\frac{1}{n})O(n^{1-r})......[1.1]$$

Xie and Zhou [9] proved for $k = 0$, if $f \in C^r[-1,1]$, $r \geq 2$,for $x \in [-1,1]$, then

$$\left|f'(x) - R'_{2n+1}(x;f)\right| = O(1)w\left(f^{(r)};\frac{1}{n}\right)O\left(n^{-r+\frac{5}{2}}\right) ....[1.2]$$

3 Problem

In this paper we study about the conditions which give the following interpolational procedure on a simultaneous approximation to a differentiable function and its first derivative. Let us assume that the set of the knots are given by:

$$-1 = v_{n-1} < ... < v_1 < v_1^* < v_0 = 1 (n \geq 2), \qquad (1)$$

where $\{v_i\}_{i=1}^{n-1}$ and $\{v_i^*\}_{i=1}^{n-1}$ are the roots of the ultraspherical polynomials $P_{n-1}^{(k)}(v)$ and $P_{n-2}^{(k)}(v)$ respectively. We now assume $a_0 = 0$,where a_0 is a real number different from the mentioned interscaled system of nodal points v_i and v_i^* We can say that here exists a unique polynomial $T_m(v)$ on the knots (1). This polynomial $T_m(v)$ has degree at most m, where $m = 2n+2k$ which

satisfies the following interpolatory conditions

$$T_m(v_i) = y_i;(i = 0,1,2,...,n-1), \qquad (2)$$

$$T'_m(v_i^*) = y'_i;(i = 1,2,\ldots,n-2), \qquad (3)$$

with the Hermite(boundary) conditions:

$$T_m^l(1) = y_1^l; (l = 0, 1, 2, \ldots, k), \quad (4)$$

$$T_m^l(-1) = y_{-1}^l; (l = 0, 1, 2, \ldots, k+1). \quad (5)$$

where y_i, y_i', y_1^l and y_{-1}^l are some arbitrary real numbers. Here k is a nonnegative fixed integer. We have given the explicit formulae for the fundamental polynomials, the existence and uniqueness of the interpolatory polynomial and the order of convergence along with convergence theorems.

4 Preliminaries

Few well known results of the Ultraspherical Polynomial[1](4.2.1) represented by $P_n^{(k)}(v)$ are given. Here $P_n^{(k)}(v) = P_n^{(k,k)}(v)$ $(k > -1, n \geq 0)$ and it satisfies the following properties:-

$$(1-v^2)P_n^{(k)''}(v) - 2v(k+1)P_n^{(k)'}(v) + n(n+2k+1)P_n^{(k)}(v) = 0, \quad (6)$$

$$P_n^{(k)'}(v) = \frac{n+2k+1}{2} P_{n-1}^{(k+1)}(v), \quad (7)$$

$$|P_n^{(k)}(v)| = O(n^k), v \in [-1, 1], \quad (8) \qquad (1-v^2)^{\frac{k}{2}+\frac{1}{4}} |P_n^{(k)}(v)| = O\left(\frac{1}{\sqrt{n}}\right). \quad (9)$$

Now, some properties regarding the fundamental polynomials of the Lagranges interpolation are as follows:-

$$l_j(v) = \frac{P_n^{(k)}(v)}{P_n^{(k)\prime}(v_j)\,(v-v_j)}, \quad (10)$$

$$l_j^*(v) = \frac{P_{n-1}^{(k+1)}(v)}{P_{n-1}^{(k+1)\prime}(v_j^*)\,(v-v_j^*)}. \quad (11)$$

Also :

$$l_j(v) = \frac{\tilde{h}_n^{(k)}}{(1-v_j^2)\left[P_n^{(k)'}(v_j)\right]^2} \sum_{\nu=0}^{n-1} \frac{1}{h_\nu^{(k)}} P_\nu^{(k)}(v_j)\, P_\nu^{(k)}(v), \quad (12)$$

where

$$\tilde{h}_n^{(k)} = \frac{2^{2k}}{\Gamma(n+1))} \frac{\Gamma^2(n+k+1))}{\Gamma(n+2k+1)} \sim k_1 \quad (13)$$

$$h_\nu^{(k)} = \frac{2^{2k+1}}{(2\nu+2k+1)} \frac{\Gamma^2(\nu+k+1))}{\Gamma(\nu+1)\Gamma(\nu+2k+1)} \begin{cases} \sim \frac{1}{\nu} & (\nu > 0) \\ = k_2 & (\nu = 0) \end{cases}, \quad (14)$$

here k_2 depends on k solely.

Also $v_1 > v_2 >$..............v_n which are the roots of $P_n^{(k)}(v)$, satisfy the following conditions:

$$(15) \qquad (1 - v_j^2) \sim \begin{cases} \frac{j^2}{n^2} & (v_j \geq 0) \\ \frac{(n-j)^2}{n^2} & (v_j < 0) \end{cases},$$

$$(16) \quad \left|P_n^{(k)\prime}(v_j)\right| \sim \begin{cases} \frac{n^{k+2}}{j^{k+3/2}} & (v_j \geq 0) \\ \frac{n^{k+2}}{(n-j)^{k+3/2}} & (v_j < 0) \end{cases}$$

5 Explicit Representation of the Interpolatory Polynomial

We say that $T_m(v)$ which satisfies (2) to (5) as:

$$T_m(v) = \sum_{j=0}^{n-1} Y_{j1}(v) y_j + \sum_{j=1}^{n-2} Y_{j2}(v) y_{j*} + \sum_{j=0}^{k} Y_{j3}(v) y_{1}^{(l)} + \sum_{j=0}^{k+1} Y_{j4}(v) y_{-1}^{(l)}, \quad (17)$$

where $Y_1(v)$ is the fundamental polynomial of the first kind, while $Y_2(u)$ is the fundamental polynomial of the second kind. $Y_3(u)$ and $Y_4(u)$ the fundamental polynomials which correspond to the Hermite (boundary) conditions, each of degree atmost $2n + 2k$. These polynomials are uniquely determined using the following conditions: for $j = 0,1,2....n - 1$

$$\begin{cases} Y_{j1}(v_i) = \delta_{ji}, & (i = 0,1,2,\ldots,n-1) \\ Y'_{j1}(v_i^*) = 0, & (i = 1,2,\ldots,n-2) \\ Y_{j1}^{(l)}(1) = 0, & (l = 0,1,\ldots,k) \\ Y_{j1}^{(l)}(-1) = 0, & (l = 0,1,\ldots,k+1) \end{cases} \quad (18)$$

for $j = 1,2....n - 2$

 $(i = 0,1,2,...,n-1)$ (19)
Y $(i = 1,2,...,n-2)$
 $Y_{j'2j}(2v(i*v) =i) = 0\delta_{ji},$ $(l = 0,1,...,k)$
$(l = 0,1,...,k+1)$
$Y_{j(2l)}(1) = 0,$

 $Y_{j(2l)}(-1) = 0,$

for $j = 0,1...k$

$$\begin{cases} Y_{j3}(v_i) = 0, & (i = 0,1,2,...,n-1) \\ Y'_{j3}(v_i^*) = 0 & (i = 1,2,...,n-2) \\ Y_{j3}^{(l)}(1) = \delta_{ji}, & (l = 0,1,...,k) \\ Y_{j3}^{(l)}(-1) = 0, & (l = 0,1,...,k+1) \end{cases} \quad (20)$$

for $j = 0,1,2,...k+1$

$$\begin{cases} Y_{j4}(v_i) = 0, & (i = 0,1,2,\ldots,n-1) \\ Y'_{j4}(v_i^*) = 0, & (i = 1,2,\ldots,n-2) \\ Y_{j4}^{(l)}(1) = 0, & (l = 0,1,\ldots,k) \\ Y_{j4}^{(l)}(-1) = \delta_{ji}, & (l = 0,1,\ldots,k+1) \end{cases} \quad (21)$$

The explicit form of the polynomials is given by the succeeding lemmas.

Lemma1: $Y_{j1}(v)$, the fundamental polynomial for $j = 0,1,2,...n-1$ that satisfies the interpolatory conditions (18) is given by:

$$Y_{j1}(v) = l_k^2(v)+c_1 P_{n-1}^k(v)\left(1-v^2\right)^{k+2}(1+x)\int_0^v \frac{l_k(t)-\left[c_2+c_3\left(t-v_i\right)P_{n-2}^k(t)\right]dt}{\left(t-v_i\right)^2}$$

(22) Where

$$c_1 = \frac{2}{P_{n-1}^{k\prime}(v_i^*)}, \quad (23)$$

and

$$c_2 = \frac{1}{P_{n-2}^k(v_i)}, \quad (24)$$

and

$$c_3 = \frac{-l_k'(v_i) + [P_{n-2}^{k\prime}(v_i)/P_{n-2}^{k}(v_i)]}{P_{n-2}^{k}(v_i) - (v - v_i)P_{n-2}^{k\prime}(v_i)} \tag{25}$$

Lemma 2: $Y_{j2}(v)$, the fundamental polynomial for $j = 1,2,...n - 1$ that satisfies the interpolatory conditions (19) is given by:

$$Y_{j2}(v) = c_4(1 - v^2)^{k+2}(1 + v)P_{n-1}^{k}(v)\int_0^v L_k(t)dt \tag{26}$$

where

$$c_4 = \frac{1}{P_{n-1}^{k}(v_i^*)(1 - v_i^{*2})^k[(k+1)(1+v_i^*) + (1 - v_i^{*2})]\int_0^{v_i^*} L_k(t)dt + (1 - v_i^{*2})(1 + v_i^*)} \tag{27}$$

Lemma 3 $Y_{j3}(v)$, the fundamental polynomial for $j = 0,1,2....k$ that satisfies the interpolatory conditions (20) is given by:

$$Y_{j3}(v) = ((1-v)^j(1+v)^k P_{n-1}^{k}(v)P_{n-2}^{k}(v)p_j(v) + \left(1 - v^2\right)^{k+1} P_{n-2}^{k}(v)g_{n+1} \tag{28}$$

where degree of $p_j(x) = k - j + 4$ and degree of $g_{n+1} = n + 1$.

Whereas g_{n+1} is given by:

$$g_{n+1} = \frac{-P_{n-1}^{k}(v_i)p_j(x)(1 + v_i)^{-1}}{(1 - v_i)^{k-j+1}} \tag{29}$$

Lemma 4 $Y_{j4}(v)$, the fundamental polynomial for $j = 0,1,2....k+1$ that satisfies the interpolatory conditions (21) is given by:

$$\begin{aligned} Y_{j4}(v) =& (1 - v)^{k+2}(1 + v)^j P_{n-2}^{k}(v)P_{n-1}^{k}(v)\bar{u}_j(x) \\ &+ \left(1 - v^2\right)^{k+2} P_{n-2}^{k}(v)\int_{-1}^{v} \frac{\left\{\bar{v}_j(t)P_{n-1}^{k}(t) - \bar{u}_j(t)P_{n-2}^{k}(t)\right\}dt}{(1+t)^{k+2-j}} \quad (j = 0, 1, \ldots, k+1) \end{aligned}$$

Here $\bar{u}_j(x)$ and $\tilde{v}_j(x)$ are uniquely determined polynomials of degree at most $k - j + 2$ and $k - j + 1$ respectively and

$$Y_{k+2}(v) = \frac{\left(1 - v^2\right)^{k+2} P_{n-1}^{(k)}(v)}{(k+2)!2^{k+2}P_{n-1}^{(k)\prime}(-1)}, \text{for } \ j = k + 2$$

6 Existence

Existence: By Lemma 1 to Lemma 4, it is evident that the polynomial $T_m(v)$ satisfies the conditions (2) to (5). So there exists an interpolatory polynomial $T_m(v)$ of degree $2n + 2k$.

7 Order of Convergence

Theorem 1: *The first derivative of the first kind of fundamental polynomial on* $[-1,1]$ *for* $n \geq 2$ *and* $k > 0$*, holds as:*

$$\sum_{j=1}^{n} \left(1-v^2\right) Y'_{j1}(v) = O\left(n^{11/2}\right). \tag{30}$$

Proof: Differentiating (22) we get:

$$\sum_{j=1}^{n-1} |Y'_{j1}(v)| = \zeta_1 + \zeta_2 + \zeta_3 \tag{31}$$

$$Y'_{j1} = 2l_j(v)l'_j(v) + c_1 P^{k\prime}_{n-1}(v)(1-v^2)^{k+2} \int_0^v \frac{l_j(t) - [c_2 + c_3(t-v_i)P^{k\prime}_{n-1}(t)dt]}{(v-v_i)^2}$$

$$+ \, c_1 P^{k}_{n-1}(v)(k+2)(-2v)(1-v^2)^{k+1} \int_0^v \frac{l_j(t)-[c_2+c_3(t-v_i)P^{k\prime}_{n-1}(t)dt]}{(v-v_i)^2}$$

$$+ \, c_1 P^{k\prime}_{n-1}(v)(1-v^2)^{k+2}\left[\frac{l_j(v)-[c_2+c_3(v-v_i)P^{k\prime}_{n-1}(v)]}{(v-v_i)^2} - \frac{l_j(0)-[c_2+c_3(-v_i)P^{k\prime}_{n-1}(0)]}{(v_i)^2}\right]. \tag{33}$$

Here

$$\zeta_1 = 2l_j(v)l'_j(v) + c_1 P^{k\prime}_{n-1}(v)(1-v^2)^{k+2} \int_0^v \frac{l_j(t) - [c_2 + c_3(t-v_i)P^{k\prime}_{n-1}(t)dt]}{(v-v_i)^2}, \tag{34}$$

$$\zeta_2 = c_1 P^{k}_{n-1}(v)(k+2)(-2v)(1-v^2)^{k+1} \int_0^v \frac{l_j(t) - [c_2 + c_3(t-v_i)P^{k\prime}_{n-1}(t)dt]}{(v-v_i)^2} \tag{35}$$

and

$$\zeta_3 = c_1 P^{k\prime}_{n-1}(v)(1-v^2)^{k+2}\left[\frac{l_j(v) - [c_2 + c_3(v-v_i)P^{k\prime}_{n-1}(v)]}{(v-v_i)^2} - \frac{l_j(0) - [c_2 + c_3(-v_i)P^{k\prime}_{n-1}(0)]}{(v_i)^2}\right]. \tag{36}$$

By using the preliminaries (8), (9), (13), (14), (15), (16) and the decomposition (12) for $l_j(v)$ we get:

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ISSN: 0030-5324
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$$\zeta_1 = o(n^{11/2}) \quad (37)$$

Similarly, we can find the values of ζ_2 and ζ_3 as:

$$\zeta_2 = o(n^{11/2}) \quad (38)$$

and

$$\zeta_3 = o(n^{9/2}). \quad (39)$$

Hence the given theorem is proved.

Theorem 2: *The first derivative of the first kind of fundamental polynomial on* $[-1,1]$ *for* $n \geq 2$ *and* $k > 0$*, holds as:*

$$\sum_{j=1}^{n} Y'_{j2}(v) = O\left(n^{9/2}\right)) \quad (40)$$

proof: Differentiating (26),we get :

$$\sum_{j=1}^{n-1} |Y'_{j2}(v)| = \eta_1 + \eta_2 + \eta_3$$

$$Y'_{j2}(v) = c_4(k+1)(1-v^2)^k(-2v)(1+v)P^k_{n-1}(v)\int_0^v L_k(t)dt \quad (41)$$

$$+c_4(1-v^2)^{k+1}P^k_{n-1}(v)\int_0^v L_k(t)dt + c_4(1-v^2)^{k+1}(1+v)P^{k\prime}_{n-1}(v)\int_0^v L_k(t)dt$$

$$+c_4(1-v^2)^{k+1}(1+v)P^k_{n-1}(v)[L_k(v) - L_k(0)]$$
(42)

where

$$\eta_1 = c_4(k+1)(1-v^2)^k(-2v)(1+v)P^k_{n-1}(v)\int_0^v L_k(t)dt + (1-v^2)^{k+1}P^k_{n-1}(v)\int_0^v L_k(t)dt, \quad (43)$$

$$\eta_2 = c_4(1-v^2)^{k+1}(1+v)P^{k\prime}_{n-1}(v)\int_0^v L_k(t)dt, \quad (44)$$

and

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Of the
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M.S. University of Baroda
ISSN: 0030-5324
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$$\eta_3 = c_4(1-v^2)^{k+1}(1+v)P_{n-1}^{k\prime}(v)\int_0^v L_k(t)dt. \quad (45)$$

By using the preliminaries (8), (9), (13), (14), (15), (16) and the decomposition (12) for $L_j(v)$ we get:

$$\eta_1 = o(n^{7/2}) \quad (46)$$

Similarly, we can find the values of η_2 and η_3 as:

$$\eta_2 = o(n^{9/2}) \quad (47)$$

and

$$\eta_3 = o(n^{9/2}). \quad (48)$$

Hence the given theorem is proved.

8 Main Theorem

Let $m = 2n+2k$ *and let* $\{v_i\}_{i=1}^{n-1}$ *and* $\{v_i^*\}_{i=1}^{n-2}$ *be the root of the Ultraspherical polynomials* $P_{n-1}^{(k)}(v)$ *and* $P_{n-2}^{(k)}(v)$ *respectively, if* $f \in C^r[-1,1]$ $(r \geq k+1, n \geq 2r-k+2)$, *then the interpolatory polynomial*

$$T_m(v;f) = \sum_{j=0}^{n-1} f(v_i)Y_{j1}(v) + \sum_{j=1}^{n-2} f'(v_i^*)Y_{j2}(v) + \sum_{j=0}^{k} f^j(1)Y_{j3}(v) + \sum_{j=0}^{k+1} f^j(-1)Y_{j4}(v) \quad (49)$$

for $x \in [-1,1]$ *satisfies*

$$|f'(x) - T_m'(v;f)| = w(f^{(r)};\frac{1}{n})O(n^{1-r+11/2}), \quad (50)$$

where the fundamental polynomials $Y_{i1}(v), Y_{i2}(v), Y_{j3}(v)$ and $Y_{j4}(v)$ are given in Lemma 1, Lemma 2, Lemma 3 and Lemma 4.

Proof. For $P_n^k(v),\ k = 0$ we refer to [7], proved by Xie and Zhou and now we prove for the case $k = 1$. Let $f \in C^r[-1,1]$, then by the theorem of Gopengauz [2] we can say for every $m > 4r+5$ there exists a polynomial $p_m(v)$ of degree at most m such that for $j = 0,...,r$

$$|f^{(j)}(v) - p_m^{(j)}(v)| \leq M_{r,j}(\frac{\sqrt{1-v^2}}{m})^{r-j} w(f^{(r)};\frac{\sqrt{1-v^2}}{m}). \quad (51)$$

where $w(f^{(r)};.)$ represents the modulus of continuity of the function $f^{(r)}(v)$. The constants $M_{r,j}$ depend on r and j only. In addition,

$f^{(j)}(\pm 1) = p_m^{(j)}(\pm 1)$ (j = 0...r).

By the uniqueness of interpolational polynomial $T_m(v;f)$, it is clear that $T_m(v;p_m) = p_m(v)$.

Hence for $x \in [-1,1]$

$$|f'(v) - T'_m(v;f)| \leq |f'(v) - p'_m(v)| + |T'_m(v;p_m) - T'_m(v;f)|$$
$$\leq |f'(v) - p'_m(v)| + \sum_{j=1}^{n-1} |f(v_j) - p_m(v_j)| \left|Y'_{j1}(v)\right| + \sum_{j=1}^{n-1} |f'(v_j) - p'_m(v_j)| \left|Y'_{j2}(v)\right|$$

Using (48) and (51) and applying the estimates (30) and (40), we obtain

$$|f'(x) - T'_m(v;f)| = w(f^{(r)};\frac{1}{n})O(n^{11/2-r}),$$

which is the proof of main theorem.

By using this main theorem and (1.1), we can give the conclusion of the convergence theorem.

9 Conclusion

Let $\{v\}_{n-1}^{(k)}$ $\{v_i^*\}_{i=1}^{n-1}$ are the roots of the ultraspherical polynomials $P_{n-1}^{(k+1)}(v)$ and $P_{n-2}^{(k+1)}(v)$ respectively. If $f \in C^{k+2}[-1,1]$, $f^{k+2} \in Lip a$, $\alpha > \frac{1}{2}$,then $T_m(v;f)$ and $T'_m(v;f)$ uniformly converge to $f(v)$ and $f'(v)$,respectively on $[-1,1]$ as $n \to \infty$ where $m = 2n+2k$ .Then there exists a polynomial $T_m(v)$ satisfying conditions (2) to (5), which is the required polynomial.

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ISSN: 0030-5324
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