Journal
Of the
Oriental Institute
M.S. University of Baroda

ISSN: 0030-5324
UGC CARE Group 1

An Interpolation Process on Laguerre Abscissas with an Additional condition

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Abstract

This paper is devoted to studying a Pál-type interpolation problem on the roots of Laguerre polynomials of degree n and its derivative of degree $n-1$. In this paper, we study an interpolation on the polynomials with an additional condition on the zeros of Laguerre polynomials. The mixed type (0,1;0)-interpolation problem is studied in a unified way. The objective of this study is to identify a single interpolatory polynomial with degree at most $3n+k$ that satisfies the interpolatory requirements. In the regular cases we find the explicit forms of the interpolational polynomials. Under certain conditions over the Laguerre polynomial and its derivative, we also obtain the estimates of the fundamental polynomials on the whole real line.

2010 Mathematics Subject Classification $41A10, 97N50$.

Keywords and phrases. Lagrange interpolation; Laguerre polynomial;, Pál-type interpolation; fundamental polynomials; explicit form; order of convergence

Journal
Of the
Oriental Institute
M.S. University of Baroda
ISSN: 0030-5324
UGC CARE Group 1

1 Introduction

Interpolation uses functions that traverse precisely through all of the input points and may be applied to modest amounts of data. Without having to pass exactly through the given points, we arrive to the function that pass through a set of data in the best way feasible using approximation. Interpolation entails the application of an interpolatory function to every point that is given, whereas approximation allows some degree of error and the resulting function can be smooth.

The interpolational problem that arise when the values of the function and its successive derivatives are specified at the provided set of points have been explored by Pál [7], Srivastava [8], Ojha [6], Lénárd [4] and Mathur [5]. Srivastava and Singh [9] [10] studied the Pál-type interpolation problem on the roots of Ultraspherical polynomials. Non-consecutive derivatives are employed in the interpolation process to extract the data while utilizing Lacunary interpolation.

On the interval $[-1,1]$ with the additional knot x_n^*, where x_n^* is equivalent to one of the nodal points x_k $(k=1,\dots n)$, Xie [14] proposed a new explicit formula of pál-type interpolation. Szili [13] looked at Pál-type interpolation on Hermite polynomials with the extra point $x_0 = 0$. Szili [12] studied the inverse Pál interpolational problem on the roots of the integrated Legendre polynomials. Eneduanya [2] investigated the special case for the Legendre polynomial.

The Pál type interpolation problem on the nodes of Laguerre abscissas was also investigated by Lénárd [3] and Chak [1]. Consider the two interscaled systems of nodal points $\{x_i\}_{i=1}^n$ and $\{x_i^*\}_{i=1}^{n-1}$, that is,

$$-\infty < x_1 < x_1^* < x_2 < \cdots < x_{n-1} < x_{n-1}^* < x_n < +\infty.$$

The function values in Pál type interpolation are specified at the zeros of $w_n(x)$, whereas the derivative values are specified at the zeros of $w_n^{'}(x)$, where

$$w_n(x) = (x-x_1)(x-x_2)...(x-x_n) \tag{1}$$

and

$$w_n^{'}(x) = n(x-x_1^*)...(x-x_{n-1}^*).$$

Laguerre polynomial $L_n^{(k)}(x)$ $(k>-1)$ has n different real roots in $(0,\infty)$ and the inter scaled system of nodal points can be used to determine the zeros of $L_n^{(k)}(x)$ and $L_n^{(k)'}(x)$. Let $\{\mu_i\}_{i=1}^n$ and $\{\mu_i^*\}_{i=1}^{n-1}$ be the two sets of nodal points in the interval $[0,\infty)$ inter scaled such that

$$0 \leqslant \mu_1 < \mu_1^* < \mu_2 < ... < \mu_{n-1} < \mu_{n-1}^* < \mu_n < +\infty. \tag{2}$$

We seek to determine a polynomial $A_n(x)$ of lowest possible degree $3n+k$ satisfying the interpolatory conditions:

$$\begin{cases} A_n(\mu_i) = \alpha_i^*, i = 1,2\dots,n, \\ A_n^{'}(\mu_i) = \beta_i^*, i = 1,2\dots,n, \\ A_n(\mu_i^*) = \gamma_i^*, i = 1,2\dots,n-1, \\ A_n^{(j)}(\mu_0) = \varphi_0^{*(j)}, j = 0,1,\dots,k, \end{cases} \tag{3}$$

and

(4) $$A_n(0) = 0,$$

where $\{\alpha_i^*\}_{i=1}^n$, $\{\beta_i^*\}_{i=1}^n$, $\{\gamma_i^*\}_{i=1}^{n-1}$, and $\{\varphi_0^{*(j)}\}_{j=0}^k$, are arbitrary real numbers. The objective is to consider the problem of explicit representation, estimation of sequence $\{A_n(x)\}$ of polynomials of degree $\leqslant 3n+k$.

2 Preliminaries

We shall use some well known properties and result, Yadav [15], of the Laguerre Polynomial $L_n^{(k)}(x)$ which are as follows: The differential equation of Laguerre polynomial is given by

(5) $$xD^2 L_n^k(x) + (1+k-x)DL_n^k(x) + nL_n^k(x) = 0,$$

where n is a positive integer and $k > -1$. For the roots of $L_n^{(k)}(x)$ we have

(6) $$2\sqrt{x_j} = \frac{1}{\sqrt{n}}[j\pi + O(1)],$$

(7) $$|L_n^{(k)'}(x_j)| \sim j^{-k-\frac{3}{2}} n^{k+1}, (0 < x_j \leq \Omega, n = 1, 2, 3, ...),$$

(8) $$|L_n^k(x)| = \begin{cases} x^{-\frac{k}{2}-\frac{1}{4}} O(n^{\frac{k}{2}-\frac{1}{4}}), & cn^{-1} \leq x \leq \Omega \\ O(n^k), & 0 \leq x \leq cn^{-1}, \end{cases}$$

(9) $$O(l_j(x)) = O(l_j^*(x)) = 1,$$

(10) $$|x - x_k| \sim \frac{k^2}{n}.$$

Now we also have some properties of fundamental polynomials of the Lagranges interpolation which are given as:

(11) $$l_j(x) = \frac{L_n^{(k)}(x)}{L_n^{(k)\prime}(x_j)\,(x-x_j)},$$

(12) $$l_j^*(x) = \frac{L_n^{(k)'}(x)}{L_n^{(k)''}(y_j)(x-y_j)}.$$

Here $l_j(x)$ and $l_j^*(x)$ are the polynomials of degree $n-1$ and $n-2$ respectively.

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Of the
Oriental Institute
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ISSN: 0030-5324
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3 Explicit Representation of Interpolatory Polynomial

Let $2n-1$ points in $(0, \infty)$ be given by (2). Then to the prescribed numbers $\{\alpha_i^*\}_{i=1}^n$, $\{\beta_i^*\}_{i=1}^n$, $\{\gamma_i^*\}_{i=1}^{n-1}$, there exists a unique polynomial $\{A_n(x)\}$ of degree $\leqslant 3n+k$ satisfying the conditions (3) and (4).

The polynomial $A_n(x)$ is explicitly given by:

$$A_n(x) = \sum_{j=0}^{n} \alpha_j^* U_j(x) + \sum_{j=1}^{n} \beta_j^* V_j(x) + \sum_{j=1}^{n-1} \gamma_j^* W_j(x) + \sum_{j=0}^{k} \varphi_0^{*(j)} C_j(x), \tag{13}$$

where $\{U_j(x)\}_{j=0}^n$, $\{V_j(x)\}_{j=1}^n$, $\{W_j(x)\}_{j=1}^{n-1}$, and $\{C_j(x)\}_{j=0}^k$ are the polynomial having the degree $\leqslant 3n+k$. These polynomials are unique and satisfies the following conditions:

For $j = 0, 1, 2, \ldots, n$,

$$\begin{cases} U_j(x_i) = \delta_{ij}, & (i = 1, 2, \ldots, n), \\ U_j'(x_i) = 0, & (i = 1, 2, \ldots, n), \\ U_j(y_i) = 0, & (i = 1, 2, \ldots, n-1), \\ U_j^l(0) = 0, & (l = 0, 1, \ldots, k), \end{cases} \tag{14}$$

for $j = 1, 2, \ldots, n$,

$$\begin{cases} V_j(x_i) = 0, & (i = 1, 2, \ldots, n), \\ V_j'(x_i) = \delta_{ij}, & (i = 1, 2, \ldots, n), \\ V_j(y_i) = 0, & (i = 1, 2, \ldots, n-1), \\ V_j^l(0) = 0, & (l = 0, 1, \ldots, k), \end{cases} \tag{15}$$

for $j = 1, 2, \ldots, n-1$,

$$\begin{cases} W_j(x_i) = 0, & (i = 1, 2, \ldots, n), \\ W_j'(x_i) = 0, & (i = 1, 2, \ldots, n), \\ W_j(y_i) = \delta_{ij}, & (i = 1, 2, \ldots, n-1), \\ W_j^l(0) = 0, & (l = 0, 1, \ldots, k), \end{cases} \tag{16}$$

for $l = 0, 1, \ldots, k$,

$$\begin{cases} C_k(x_i) = 0, & (i = 1, 2, \ldots, n), \\ C_k'(x_i) = 0, & (i = 1, 2, \ldots, n), \\ C_k(y_i) = 0, & (i = 1, 2, \ldots, n-1), \\ C_k^l(0) = \delta_{ij}, & (l = 0, 1, \ldots, k), \end{cases} \tag{17}$$

Here δ_{ij} is a Kronecker delta,

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} . \tag{18}$$

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ISSN: 0030-5324
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The explicit form of the $U_j(x), V_j(x), W_j(x)$ and $C_k(x)$ are given in the following theorem.

Lemma 1:

The fundamental polynomial $\{U_j(x)\}_{j=0}^{n}$ satisfying the interpolatory conditions (14) is given by:

For $j = 0$,

$$U_0(x) = \frac{L_n^{(k)'}(x)\left[L_n^{(k)}(x)\right]^2}{L_n^{(k)'}(0)\left[L_n^{(k)}(0)\right]^2}, \tag{19}$$

For $j = 1, 2....n$,

$$U_j(x) = \frac{x^{k+2}[l_j(x)]^2 L_n^{(k)'}(x)}{x_j^{k+2} L_n^{(k)'}(x_j)}\left[1 + \frac{(k-2x_j)}{x_j}(x - x_j)\right], \tag{20}$$

where $l_j(x)$ is given by (11).

Proof:

For $j = 1, 2....n$, let

$$U_j^*(x) = U_1 x^{k+2}[l_j(x)]^2 L_n^{(k)'}(x) + U_2 x^{k+2}(x - x_j)[l_j(x)]^2 L_n^{(k)'}(x), \tag{21}$$

be a polynomial of degree $\leqslant 3n + k$. Note that $U_j^*(x)$ satisfies the equations (14) provided

$$U_1 = \frac{1}{x_j^{k+2} L_n^{(k)'}(x_j)}. \tag{22}$$

Also for $j = 1, 2....n$, the condition $U_j^{*'}(x_i) = 0$ provides

$$U_2 = \frac{(k-2x_j)}{x_j^{k+3} L_n^{(k)'}(x_j)}. \tag{23}$$

Thus,

$$U_j^*(x) \equiv U_j(x),$$

which completes the proof of the lemma.

Lemma 2:

The fundamental polynomial $\{V_j(x)\}_{j=1}^{n-1}$ satisfying the interpolatory conditions (15) is given by:

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ISSN: 0030-5324
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For $j = 1, 2....n-1$,

$$V_j(x) = \frac{x^{k+2} l_j(x) L_n^{(k)}(x) L_n^{(k)'}(x)}{x_j^{k+2} \left[L_n^{(k)'}(x_j) \right]^2}, \tag{24}$$

where $l_j(x)$ is given by (11).

Proof:

For $j = 1, 2....n$, let

$$V_j^*(x) = v_1 x^{k+2} l_j(x) L_n^{(k)}(x) L_n^{(k)'}(x) \tag{25}$$

be a polynomial of degree $\leqslant 3n+k$. Note that $V_j^*(x)$ satisfies the equations (15) provided

$$v_1 = \frac{1}{x_j^{k+2} \left[L_n^{(k)'}(x_j) \right]^2}, \tag{26}$$

Thus,

$$V_j^*(x) \equiv V_j(x),$$

which completes the proof of the lemma.

Lemma 3:

The fundamental polynomial $\{W_j(x)\}_{j=1}^{n-1}$ satisfying the interpolatory conditions (16) is given by:

For $j = 1, 2....n-1$,

$$W_j(x) = \frac{x^{k+2} l_j^*(x) \left[L_n^{(k)}(x) \right]^2}{y_j^{k+2} \left[L_n^{(k)}(y_j) \right]^2}, \tag{27}$$

where $l_j^*(x)$ is given by (12).

Proof:

For $j = 1, 2....n-1$, let

$$W_j^*(x) = w_1 x^{k+2} l_j^*(x) \left[L_n^{(k)}(x) \right]^2 \tag{28}$$

be a polynomial of degree $\leqslant 3n+k$. Note that $W_j^*(x)$ satisfies the equations (16) provided

$$w_1 = \frac{1}{y_j^{k+2} \left[L_n^{(k)}(y_j) \right]^2}, \tag{29}$$

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ISSN: 0030-5324
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Thus,

$$W_j^*(x) \equiv W_j(x),$$

it completes the lemma's proof.

Lemma 4:

The interpolatory conditions (17) are satisfied by the basic polynomial $\{C_j(x)\}_{j=0}^{k}$ which is provided by:

For $j = 0, 1, 2, ..., k-1$,

$$C_j(x) = a_j(x)x^{j+1}[L_n^{(k)'}(x)]^2 L_n^{(k)}(x) + x^{k+1} L_n^{(k)}(x) L_n^{(k)'}(x)$$

$$\left[c_j^* - \frac{L_n^{(k)'}(x)a_j(x) + b_j(x)L_n^{(k)}(x)}{x^{k-j}}\right], \tag{30}$$

and

$$C_k(x) = \frac{1}{k! L_n^{(k)}(0)[L_n^{(k)'}(0)]^2} x^{k+1}[L_n^{(k)}(x)]^2 L_n^{(k)'}(x), \tag{31}$$

where the degree of the polynomial $a_j(x)$ is at most $(k-j-1)$ and the degree of the polynomial $b_j(x)$ is at most $(k-j)$.

Proof:

To find $C_j(x)$, let us suppose that $C_j(x)$ for some fixed $j \in \{0, 1, ..., k-1\}$

$$C_j^*(x) = a_j^*(x)x^{j+1}[L_n^{(k)'}(x)]^2 L_n^{(k)}(x) + x^{k+1} L_n^{(k)}(x) L_n^{(k)'}(x) b_n^*(x), \tag{32}$$

where the degree of the polynomial $a_j^*(x)$ is $(k-j-1)$ and the degree of the polynomial $b_n^*(x)$ is n. Also note that for $(l = 0, 1, ..., j-1)$, $C_j^{(l)}(0) = 0$. We know that $L_n^{(k)}(x_i) = 0$ and $L_n^{(k)'}(y_i) = 0$, therefore $C_j(x_i) = 0$ and $C_j(y_i) = 0$ for $i = 1, 2, ..., n$. The coefficient of the polynomial $a_j^*(x)$, for $(l = j, ..., k-1)$, is calculated by:

$$C_j^{(l)}(0) = \frac{d^l}{dx^l}\left[a_j^*(x)x^j[L_n^{(k)'}(x)]^2 L_n^{(k)}(x)\right]_{x=0} = \delta_{ij}. \tag{33}$$

Now using the condition $C_j'(x_i) = 0$ of (17), we will have

$$b_n^*(x_i) = -(x_i)^{j-k} L_n^{(k)'}(x_i) a_j^*(x_i),$$

this will imply the value of $b_n^*(x)$ as:

$$b_n^*(x) = -\frac{L_n^{(k)}(x)a_j^*(x) + b_j^*(x)L_n^{(k)'}(x)}{x^{k-j}}, \tag{34}$$

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where the degree of the polynomial $a_j^*(x)$ is at most $(k-j-1)$ and the degree of the polynomial $b_j^*(x)$ is at most $(k-j)$. By using the equations (32) and (34) we get the polynomial $C_j(x)$ of degree $\leq 3n+k$ holding the equations (17).

Now we state our main theorem.

Theorem 1:

Assuming that the interpolatory function $f : R \to R$ is continuous as well as differentiable such that

$C(m) = \{f(x) : f(x) = O(x^m) \ as \ x \longrightarrow \infty;\}$,

where m is a non negative integer, f is continuous function in the interval $[0, \infty)$, then for each $f \in C(m)$ and a non negative k,

$$(35) \qquad A_n(x) = \sum_{j=0}^{n} \alpha_j^* U_j(x) + \sum_{j=1}^{n} \beta_j^* V_j(x) + \sum_{j=1}^{n-1} \gamma_j^* W_j(x) + \sum_{j=0}^{k} \varphi_0^{*(j)} C_j(x),$$

satisfies the relation:

$$(36) \qquad |A_n(x) - f(x)| = \mathrm{O}(1)\omega\left(f, \frac{\log n}{\sqrt{n}}\right), \qquad \text{for } 0 \leq x \leq cn^{-1}$$

$$(37) \qquad |A_n(x) - f(x)| = \mathrm{O}(1)\omega\left(f, \frac{\log n}{\sqrt{n}}\right), \qquad \text{for } cn^{-1} \leq x \leq \Omega$$

here ω represents the modulus of continuity.

Prior to the proving of theorem 1, first estimate the values of the following fundamental polynomials which are listed below:

4 Estimation of the Fundamental Polynomials

Theorem 2:

Let us assume the elementary polynomial $U_j(x)$, for $j = 0, 1, 2, ..., n$ be presented by:

$$(38) \qquad U_j(x) = \frac{x^{k+2}[l_j(x)]^2 L_n^{(k)'}(x)}{x_j^{k+2} L_n^{(k)'}(x_j)} \left[1 + \frac{(k-2x_j)}{x_j}(x - x_j)\right],$$

then we have

$$(39) \qquad \sum_{j=0}^{n} |U_j(x)| = \mathrm{O}(1), \qquad \text{for } 0 \leq x \leq cn^{-1}$$

$$(40) \qquad \sum_{j=0}^{n} |U_j(x)| = \mathrm{O}(1), \qquad \text{for } cn^{-1} \leq x \leq \Omega.$$

Proof:

Rewrite the polynomial as:

$$U_j(x) = \frac{x^{k+2}[l_j(x)]^2 L_n^{(k)'}(x)}{x_j^{k+2} L_n^{(k)'}(x_j)} + \frac{(k-2x_j)}{x_j} \frac{x^{k+2}[l_j(x)]^2 L_n^{(k)'}(x)}{x_j^{k+2} L_n^{(k)'}(x_j)} (x - x_j),$$

then we have

$$\sum_{j=1}^{n} |U_j(x)| \leq \sum_{j=1}^{n} \frac{\left|x^{k+2}\right| [l_j(x)]^2 \left|L_n^{(k)'}(x)\right|}{\left|x_j^{k+2}\right| \left|L_n^{(k)'}(x_j)\right|}$$

$$(41) \qquad + \sum_{j=1}^{n} \frac{|(k-2x_j)| \left|x^{k+2}\right| [l_j(x)]^2 \left|L_n^{(k)'}(x)\right| |(x-x_j)|}{|x_j| \left|x_j^{k+2}\right| \left|L_n^{(k)'}(x_j)\right|}.$$

As $U_j(x)$ is independent to $L_n^{(k)}(x)$ and by using the equations (6), (7), (9), (10), we get the desired result.

$\sum_{j=0}^{n} |U_j(x)| = \mathrm{O}(1)$, for $0 \leq x \leq \Omega$.

Theorem 3:

Let us assume the basic polynomial $V_j(x)$, for $j = 1, 2, ..., n$ be presented by:

$$(42) \qquad V_j(x) = \frac{x^{k+2} l_j(x) L_n^{(k)}(x) L_n^{(k)'}(x)}{x_j^{k+2} \left[L_n^{(k)'}(x_j)\right]^2},$$

then we have

$$(43) \qquad \sum_{j=1}^{n} |V_j(x)| = \mathrm{O}(n^{-1}), \qquad \text{for } 0 \leq x \leq cn^{-1}$$

$$(44) \qquad \sum_{j=1}^{n} |V_j(x)| = \mathrm{O}(n^{-1}), \qquad \text{for } cn^{-1} \leq x \leq \Omega.$$

Proof:

From the polynomial $V_j(x)$ we have

$$\sum_{j=0}^{n} |V_j(x)| \leq \sum_{j=0}^{n} \frac{\left|x^{k+2}\right| |l_j(x)| \, |L_n^{(k)}(x)| |L_n^{(k)'}(x)|}{\left|x_j^{k+2}\right| \left[L_n^{(k)'}(x_j)\right]^2},$$

by using the equations (6), (7), (8), (9), we get the desired result.

$\sum_{j=1}^{n} |V_j(x)| = \mathrm{O}(n^{-1})$, for $0 \le x \le \Omega$.

Theorem 4:

Let us suppose the elementary polynomial $W_j(x)$, for $j = 1, 2, ..., n-1$ be presented by:

$$W_j(x) = \frac{x^{k+2} l_j^*(x) \left[L_n^{(k)}(x)\right]^2}{y_j^{k+2} \left[L_n^{(k)}(y_j)\right]^2}, \tag{45}$$

then we have

$$\sum_{j=1}^{n-1} |W_j(x)| = \mathrm{O}(1), \qquad \text{for } 0 \le x \le cn^{-1} \tag{46}$$

$$\sum_{j=1}^{n-1} |W_j(x)| = \mathrm{O}(1), \qquad \text{for } cn^{-1} \le x \le \Omega. \tag{47}$$

Proof:

From the polynomial $W_j(x)$ we have

$$|W_j(x)| \le \frac{\left|x^{k+2}\right| |l_j^*(x)| \left[L_n^{(k)}(x)\right]^2}{\left|y_j^{k+2}\right| \left[L_n^{(k)}(y_j)\right]^2},$$

$$\sum_{j=1}^{n-1} |W_j(x)| \le \sum_{j=1}^{n-1} \frac{\left|x^{k+2}\right| |l_j^*(x)| \left[L_n^{(k)}(x)\right]^2}{\left|y_j^{k+2}\right| \left[L_n^{(k)}(y_j)\right]^2},$$

by using the equations (6), (8), (9), we get the desired result.

$\sum_{j=1}^{n-1} |W_j(x)| = \mathrm{O}(1)$, for $0 \le x \le \Omega$.

Remark:

$C(m) = \{f(x) : f$ is continuous in $[0, \infty), f(x) = O(x^m)$ as $x \longrightarrow \infty; \}$, where $m \ge 0$ is an integer. Then by the result of Szegö [11],

$$lim_{n \longrightarrow \infty} |f(x) - H_n^{(\alpha)}(f, x)|_I = 0,$$

where $I \subset (0, \infty)$ for $\alpha \ge 0$, or $I \subset (0, \infty)$ for $-1 < \alpha < 0$. Also note that there is a function in $C(m)$ such that $\{H_n^{(\alpha)}(f, x)\}$ diverges for $\alpha \ge 0$ at $x = 0$. And for the convergence rate we have:

$$\left| f(x) - H_n^{(\alpha)}(f, x) \right|_I = \begin{cases} O\left(\omega\left(f, n^{-1-\alpha}\right)\right); & -1 < \alpha < 0 \\ O\left(\omega\left(f, \frac{logn}{\sqrt{n}}\right)\right); & \alpha \ge -\frac{1}{2}. \end{cases} \tag{48}$$

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Of the
Oriental Institute
M.S. University of Baroda

ISSN: 0030-5324
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5 Proof of the main theorem 1:

Let us suppose that $P_n(x)$ be a polynomial of degree $\leq 3n+k$ and $A_n(x)$ be given by (13). Note that $A_n(x)$ is exact for every fundamental polynomials of degree $\leq 3n+k$ therefore,

(49)

$$P_n(x) = \sum_{j=0}^{n} P_n(x_j)U_j(x) + \sum_{j=1}^{n} P_n^{'}(x_j)V_j(x) + \sum_{j=1}^{n-1} P_n(y_j)W_j(x) + \sum_{j=0}^{k} P_n(x_0)C_j(x),$$

from equation (13) and (49) we get

$$|f(x) - A_n(x)| \leq |f(x) - P_n(x)| + |P_n(x) - A_n(x)| \tag{50}$$

$$\leq |f(x) - P_n(x)| + \sum_{j=0}^{n} |f(x_j) - P_n(x_j)|\, |U_j(x)|$$
$$+ \sum_{j=1}^{n} \left| f^{'}(x_j) - P_n^{'}(x_j) \right| |V_j(x)|$$
$$+ \sum_{j=1}^{n-1} |f(y_j) - P_n(y_j)|\, |W_j(x)|$$
$$+ \sum_{j=0}^{k} \left| f^{l}(x_0) - P_n^{l}(x_0) \right| |C_j(x)|\,.$$

Thus, equation (48) and the conclusions of theorem 2, 3, and 4 complete the proof of the theorem 1.

6 Conclusion

Let $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n-1}$ be the roots of Laguerre polynomials $L_n^{(k)}(x)$ and its derivative $L_n^{(k)'}(x)$ respectively. If $f : R \to R$ be continuously differentiable interpolatory function, then there is a polynomial $A_n(x)$ having the degree at most $3n+k$ holding the equations (3) and an additional condition (4) which converges uniformly to $f(x)$ on real number line for the large value of n.

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