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# ANALYSIS AND APPLICATIONS OF BICOMPLE*X* NUMBER AND FUNCTIONAL ANALYTIC STRUCTURE OF $C_2$

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**ABSTRACT**

I propose to unveil before you some of the magnificent properties and useful concepts of the theory of bicomplex numbers and functions of a bicomplex variable. I also proposed to present certain glimpses of its multifaceted applications. Allow me to start with a brief historical background that led to the origin of this theory.

**INTRODUCTION**

The basic hurdle in the development of the subject was the concept of algebra. The only algebra known until then was the algebra of real numbers. Moreover, with Gauss' proof of its fundamental theorem, the subject "algebra" became synonymous with "study of polynomials". The nineteenth century saw two major breakthroughs in this approach.

First, the thrust of the subject was shifted from the study of polynomials to the study of the structure of algebraic system. A major step in this direction was the invention of Symbolic Algebra by the English mathematician George Peacock (1791-1858). In 1830, he published his book "Treatise of Algebra" and put forward the logical structure of an abstract algebra.

The second major breakthrough was the discovery of algebraic systems, which satisfy not all but most of the properties of the algebra of real numbers. In 1833, the Irish mathematician, Sir William Rowan Hamilton (1805-1865) developed an algebra of real numbers, which is the present day algebra of comple*x* numbers. It was the beginning of theory of algebras different from the algebra of real numbers.

Many special algebras came up since then, most noteworthy amongst them being Matri*x* algebras developed by Arthur Cayley (1821-1895) and Clifford algebra developed by William Kingdon Clifford (1845-1879).

A quaternion is a number of the type

$$\boldsymbol{X} = x_0 + ix_1 + jx_2 + kx_3$$

where $x_p \in$ R, $p$ = 0, 1, 2, 3 and $i, j, k$ are symbols such that

$$i^2 = j^2 = k^2 = -1 \text{ and } ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The addition of quaternions and multiplication by a real scalar was defined as what we call coordinatewise. The multiplication of two quaternions was defined in a typical manner (due to the hypotheses made about the symbols $i$, $j$ and $k$ as mentioned above) as

$$(x_0 + ix_1 + jx_2 + kx_3) \times (y_o + iy_1 + jy_2 + ky_3)$$

$$= (x_o y_o - x_1 y_1 - x_2 y_2 - x_3 y_3) + i(x_o y_1 + x_1 y_o + x_2 y_3 - x_3 y_2)$$

$$+ j(x_o y_2 - x_1 y_3 + x_2 y_o + x_3 y_1) + k(x_0 y_3 + x_1 y_2 - x_2 y_1 + x_3 y_0)$$

Hamilton developed the quaternion algebra by the method of trial and error. Apparently, he was looking for a multiplication in which every non-zero element must possess an inverse. Today we know that quaternions are the only 4-D division algebra. This is an evidence how, at times, the method of trial and error may lead to a beautiful as well as useful concept.

Very few of us know that modern Vector Analysis has originated from this concept of quaternions. Hamilton proposed to distinguish "$x_0$" as the "scalar part" and "$\boldsymbol{x} = ix_1 + jx_2 + kx_3$" as the "vector part" of the number. In particular, the product of two "vector parts" of quaternions took the shape

$$x\text{y} = (ix_1 + jx_2 + kx_3) \times (iy_1 + jy_2 + ky_3)$$

$$= (-x_1 y_1 - x_2 y_2 - x_3 y_3) + \{i(x_2 y_3 - x_3 y_2) + j(x_3 y_1 - x_1 y_3) + k(x_1 y_2 - x_2 y_1)\}$$

which, in modern language of vector analysis, can be put as

$$= -x\,\text{y} + (x \times \text{y})$$

The terms "scalar product" and "vector product" emerged from here only and were proposed by Hamilton himself. The contemporary workers in this area saw the possibility of using these numbers as four-dimensional vectors. Hamilton himself knew about this possibility and hoped that some day it could be used for introducing time as the fourth independent dimension and then the theory would become a powerful tool not only for geometers but also for physicists.

However, as luck would have it, the idea faced a stiff opposition from the conservative class of that era. Most surprisingly, the idea of a four-dimensional quantity was not acceptable to the engineers and scientists of the time. Especially, the idea of treating time as the fourth independent coordinate was totally dismissed.

Nevertheless, the concept had a charm which could not be thrust aside so easily. Around 1900, Josiah Willard Gibbs (1839-1903) of America and Oliver Heaviside (1850-1925) of Britain presented a camouflaged form of quaternions so that all vectors were restricted to a dimension less than or equal to three. In order to retain closure under cross product they assumed, without any logic behind the assumption, that $i \times i = j \times j = k \times k = 0$.

They cleverly renamed this camouflaged version of quaternion analysis as Vector Analysis. Surprisingly, the scientists and the engineers readily accepted this version and vector analysis attained the position where it is today, leaving its first ancestor-quaternion analysis-far behind.

Every student of algebra has learnt about quaternions as a counter example of a division ring, which is not a field. In fact, as seen above, multiplication of quaternions is not commutative. This was a very big drawback in the theory of quaternions and probably the biggest hurdle in its prosperity.

In 1892, Corrado Segre (1860-1924) [S1] came up with the concept of Multicomplex Numbers. He defined a bicomplex number as

$$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1+i_1x_2) + i_2(x_3 + i_1x_4) = z_1+i_2z_2$$

where $x \in$ R, $1 \le k \le 4$; $z_1, z_2 \in$ C and $i_1^2 = i_2^2 = -1; \; i_1i_2 = i_2i_1$

A tricomplex number was defined as a number of the type $\xi + i_3\eta, \xi, \eta$ being bicomplex numbers and $i_3^2 = -1, i_3i_2 = i_2i_3, i_3i_1 = i_1i_3$. Iteratively, an n-complex number is a number $\xi + i_n\eta, \xi, \eta$ being (n–1)-complex numbers and $i_n^2 = -1, i_ni_k = i_ki_n, \; 1 \le k \le n-1$.

The set of bicomplex, tricomplex,... numbers are denoted $C_2$, $C_3$, and so on. The set of complex numbers and real numbers are denoted $C_1$ and $C_0$, respectively. For the sake of brevity, I will confine my deliberations to bicomplex version of the theory only.

Segre wished to develop a field structure on $C_2$ but Frobenius [F1] had already shown, in 1877, that no such field could exist. Segre developed algebra of bicomplex numbers by defining coordinatewise addition and real scalar multiplication. Multiplication of bicomplex numbers was defined on the lines of Hamilton incorporating the new hypotheses about $i_1$ and $i_2$. Thus if

$$\eta = y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4 = w_1 + i_2w_2,$$

$$\begin{aligned}\xi.\eta &= (x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4) + i_1(x_1y_2 + x_2y_1 - x_3y_4 - x_4y_3) \\ &+ i_2(x_1y_3 - x_2y_4 + x_3y_1 - x_4y_2) + i_1i_2(x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1) \\ &= (z_1w_1 - z_2w_2) + i_2(z_1w_2 + z_2w_1)\end{aligned}$$

Equipped with these compositions, $C_2$ becomes a commutative algebra, which is not a field. In fact, the algebraic structure of $C_2$ differs from that of $C_1$ in many respects. I will mention only a few of them that are required in our further discussions.

### 1.1 NON-ZERO SINGULAR (NON-INVERTIBLE) ELEMENTS EXIST IN $C_2$

Actually, a bicomplex number $\xi = z_1 + i_2 z_2$ is singular if and only if $|z_1^2 + z_2^2| = 0$. Due to the existence of singular elements, the division by a bicomplex number and the cancellation laws are restricted to non-singular bicomplex numbers.

Many of us might lose interest in the subject due to the presence of the singular elements. However, as it will turn out, the set of all singular elements has an interesting and useful structure and a prominent role in the development of the theory. To emphasize this, we shall touch this topic again, a little later.

### 1.2 NON-TRIVIAL IDEMPOTENT ELEMENTS EXIST IN $C_2$

Besides 0 and 1 there are two non-trivial idempotent elements in $C_2$, denoted $e_1$ and $e_2$ and defined as

$$e_1 = (1 + i_1 i_2)/2, \qquad e_2 = (1 - i_1 i_2)/2$$

Note that $e_1 e_2 = e_2 e_1 = 0$. Further, every element of $C_2$ can be written as

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$$

Such representation of a bicomplex number as a combination of complex multiples of $e_1$ and $e_2$ is unique and is known as the Idempotent Representation of and the complex coefficients $z_1 - i_1 z_2$ and $z_1 + i_1 z_2$ are called the idempotent components of the bicomplex number $\xi = z_1 + i_2 z_2$.

The theory of bicomplex numbers and functions of a bicomplex variable gets a new tool in the shape of idempotent representation, which helps a lot in understanding and interpreting certain aspects.

The algebraic structure of $C_2$ is consistent with the idempotent representation in the sense that a binary composition of bicomplex numbers is equivalent to the corresponding binary composition of their respective idempotent components. Of particular interest are the following identities:

$$\xi^n = (z_1 + i_2 z_2)^n = (z_1 - i_1 z_2)^n e_1 + (z_1 + i_1 z_2)^n e_2$$

and $$\xi / n = \{(z_1 + i_2 z_2)/(w_1 + i_2 w_2)\}$$

$$= \{(z_1 - i_1 z_2)/(w_1 - i_1 w_2)\}e_1 + \{(z_1 + i_1 z_2)/(w_1 + i_1 w_2)\}e_2$$

Of course, $\eta$ is non-singular.

An immediate suggestion from the idempotent representation is the following definition of Complex Auxiliary Spaces $A_1$ and $A_2$, viz.,

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$$A_1 = \{z_1 - i_1 z_2;\ \ z_1, z_2 \in C_1\}$$

and $$A_2 = \{z_1 - i_1 z_2;\ \ z_1, z_2 \in C_1\}$$

Obviously, $A_1$ and $A_2$ are only different representations of $C_1$ and there is a natural homomorphism between $C_2$ and $A_1 \times A_2$. However, they have their own important place in the theory.

To illustrate this, recall that a non zero bicomple*x* number $\xi = z_1 + i_2 z_2$ is singular iff $|z_1^2 + z_2^2| = 0$. This is equivalent to $(z_1 - i_1z_2)\ (z_1 + i_1z_2) = 0$. Now, $C_1$, being a field, is devoid of zero divisors. Hence $(z_1 - i_1z_2)(z_1 + i_1z_2) = 0$ is possible if and only if one of the two comple*x* factors is zero (both the factors cannot vanish simultaneously, since $\xi \neq 0$). In other words, a bicomple*x* number is singular if and only if one of its idempotent components vanishes, i.e., if and only if the bicomple*x* number is a comple*x* multiple of $e_1$ or $e_2$.

Thus, if we define

$$I_1 = \{\xi \in C_2;\ \xi = w_1 \cdot e_1,\ w_1 \in C_1\} = \{\xi \cdot e_1;\ \xi \in C_2\}$$

$$= \{(z_1 - i_1 z_2) \cdot e_1;\ z_1, z_2 \in C_1\}$$

and

$$I_2 = \{\xi \in C_2;\ \xi = w_2 \cdot e_2,\ w_2 \in C_1\} = \{\xi \cdot e_2;\ \xi \in C_2\}$$

$$= \{(z_1 - i_1 z_2) \cdot e_1;\ z_1, z_2 \in C_1\}$$

we see that $I_1 \cap I_2 = \{0\}$ and their union $\theta_2 = I_1 \cup I_2$ is precisely the set of all singular elements of $C_2$. At times, therefore, it is convenient to realize that if $\theta_2$ is taken as the "zero element" of $C_2$, $C_2$ may be regarded as "field". Incidentally, $I_1$ and $I_2$ are principal ideals in $C_2$ generated by $e_1$ and $e_2$, respectively.

## FUNCTIONAL ANALYTIC STRUCTURE OF $C_2$

### 2.1 The Norm in $C_2$ is defined as

$$\|\xi\| = \{|z_1|^2 + |z_2|^2\}^{1/2} = \{x_1^2 + x_2^2 + x_3^2 + x_4^2\}^{1/2} = \{(|z_2 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2)/2\}^{1/2}$$

With this norm and the coordinatewise addition and scalar multiplication, $C_2$ becomes a Real Banach Space. However, it could not become a Classical Banach Algebra because the best estimate for the norm of the product could only be $\|\xi \cdot \eta\| \leq \sqrt{2} \cdot \|\xi\| \cdot \|\eta\|$ | instead of $\|\xi \cdot \eta\| \leq \|\xi\| \cdot \|\eta\|$, as in classical Banach algebras. With this shortcoming, $C_2$ is regarded as Modified Banach Algebra.

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## 2.2 BICOMPLE*X* SEQUENCES AND THEIR CONVERGENCE

Corresponding to various representations of a bicomple*x* number described above, a bicomple*x* sequence $\{\xi_n\}$, where

$$\xi_n = z_{1n} + i_2 z_{2n} = x_{1n} + i_1 x_{2n} + i_2 x_{3n} + i_1 i_2 x_{4n} = (z_{1n} - i_1 z_{2n})e_1 + (z_{1n} + i_1 z_{2n})e_2$$

can be considered as made up of any of the following:

(a) four sequences $\{x_{kn})$ in $C_0$, $1 \le k \le 4$;

(b) two sequences $(z_{kn}\}$ in $C_1$, $1 \le k \le 2$;

(c) two sequences $\{z_{1n} - i_1 z_{2n}\}$ and $\{z_{1n} + i_1 z_{2n}\}$ in $A_1$ and $A_2$, respectively. Thus the convergence of the sequence $(\xi_n)$ is equivalent to the convergence of all the sequences of any one of the above types. In fact, we have

**RESULT I.** If

$$\xi = z_1 + i_2 z_2 = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2,$$

the following four statements are equivalent:

(a) The sequence $(\xi_n)$ converges to $\xi$.

(b) The sequences $\{x_{in}\}$ converge to $x_k$, $1 \le k \le 4$.

(c) The sequences $(z_{kn})$ converge to $z_k$, $1 \le k \le 2$.

(d) The sequence $\{z_{1n} - i_1 z_{2n}\}$ converges to $z_1 - i_1 z_2$ and the sequence $\{z_{1n} + i_1 z_{2n}\}$ converges to $z_1 + i_1 z_2$.

## 2.3 BICOMPLE*X* SERIES AND THEIR CONVERGENCE

Similar discussions for a bicomple*x* series and the definition of convergence of a bicomple*x* series as that of the sequence of its partial sums yields an analogue of this result, viz.,

**RESULT II.** The following statements are equivalent:

(a) The series $\Sigma \xi_n$ converges to $\xi$

(b) The series $\Sigma x_{kn}$ converge to $x_k$, $1 \le k \le 4$.

(c) The series $\Sigma z_{kn}$ converge to $z_k$, $1 \le k \le 2$.

(d) The series $\Sigma(z_{1n} - i_1 z_{2n})$ converges to $z_1 - i_1 z_2$ and the series $(z_{1n} + i_1 z_{2n})$ converges to $z_1 + i_1 z_2$.

The summation runs from 0 to $\infty$.

### 3.1 BICOMPLEX ANALYSIS

We start with a few definitions.

**DEFINITION A:** The Cartesian set $X$, in $C_2$, determined by $X_1 \subset A_1$, and $X_2 \subset A_2$ is defined as

$$x = \{\alpha \in C_2 : \alpha = w_1 e_1 + w_2 e_2,\ (w_1, w_2) \in X_1 \times X_2\}$$

**DEFINITION B:** The Open Ball $\mathbf{B}(\alpha, r)$ with the bicomplex number a as centre and radius $r$ is defined as

$$B(\alpha, r) = \{\zeta \in C_2;\ \|\zeta - \alpha\| < r\}$$

Closed Ball $\bar{B}(\alpha, r)$ is defined analogously.

**DEFINITION C:** The Open Discus with centre s $\alpha = a + i_2 b$ and radii $r_1$ and $r_2$ is defined as

$$D(\alpha; r_1, r_2) = \{\zeta \in C_2 : |(z_1 - i_1 z_2) - (a - i_1 b)| < r_1; |(z_1 + i_1 z_2) - (a + i_1 b)| < r_2\}$$

Closed discus $\bar{D}(\alpha; r_1, r_2)$ is defined similarly.

In some cases, it is convenient to take the centre at origin and the two radii equal to **r**. Such a disc is called the Natural $C_2$ disc of radius $r$.

**RESULT III.** If $0 < r_1 \leq r_2$, then

$$B(\alpha, r_1/\sqrt{2}) \subset D(\alpha; r_1, r_2) \subset B(\alpha, \{(r_1^2 + r_2^2)/2)\}^{1/2});$$

$$\bar{B}(\alpha, r_1/\sqrt{2}) \subset \bar{D}(\alpha; r_1, r_2) \subset \bar{B}(\alpha, \{(r_1^2 + r_2^2)/2)\}^{1/2}).$$

All the inclusions are proper.

Few properties of these structures are noteworthy:

(A) A set $X$ is open in $C_2$ if and only if for each $\xi = z_1 + i_2 z_2 \in X$, there exists a discus $D(\xi; r_1, r_2)$ in $X$.

(B) A set $X$ is arcwise connected if and only if each two points in $X$ can be connected by a polygonal arc within $X$.

(C) If the Cartesian set $X$ is a domain in $C_2$, $X_1$ and $X_2$ are domains in $A_1$ and $A_2$, respectively.

(D) If $X_1$ and $X_2$ are domains in $A_1$ and $A_2$, respectively, the Cartesian set $X$ in $C_2$, determined by $X_1$ and $X_2$, is a domain in $C_2$.

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